THE CONVERGENCE OF SPECTRAL METHODS FOR NONLINEAR CONSERVATION LAWS*

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Abstract

We discuss the convergence of Fourier method for scalar nonlinear conservation laws which exhibit spontaneous shock discontinuities. Numerical tests indicate that the convergence may (and in fact in some cases must) fail, with or without post-processing of the numerical solution. Instead, we introduce here a new kind of spectrally accurate vanishing viscosity to augment the Fourier approximation of such nonlinear conservation laws. Using compensated compactness arguments, we show that this spectral viscosity prevents oscillations and convergence to the unique entropy solution follows.

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1. Introduction.

In this paper we study the convergence of spectral methods for nonlinear conservation laws. Specifically, we consider what is accepted by now as the universal model problem for such scalar laws, namely, the inviscid Burgers' equation

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}(\frac{u^2(x,t)}{2})=0 \tag{1.1}$$

subject to given initial data $u(x,t=0)$. Among the basic features of solutions to this problem [6], we recall that they may develop spontaneous jump discontinuities (shock waves) and hence the class of weak solutions must be admitted; that within this class, there are many possible solutions; and that in order to single out the unique 'physically relevant' solution among them, (1.1) is augmented with an additional entropy condition which requires

$$\frac{\partial}{\partial t}(\frac{u^2(x,t)}{2})+\frac{\partial}{\partial x}(\frac{u^2(x,t)}{3})\le 0\,. \tag{1.2}$$

The existence of physically relevant shock waves in the solution is reflected by the strict (distributional) inequality in (1.2).

We want to solve the 2π-periodic problem (1.1), (1.2) by the spectral-Fourier method. To this end, we approximate the spectral-Fourier projection of $u(x,t)$,

$$P_N u(x,t)=\sum_{k=-N}^{N}\hat{u}(k,t)e^{ikx},\qquad \hat{u}(k,t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(x,t)e^{-ikx}dx\,, \tag{1.3}$$

by an N-trigonometric polynomial, $v_N(x,t)$,

$$v_N(x,t)=\sum_{k=-N}^{N}\hat{v}_k(t)e^{ikx}\,. \tag{1.4}$$

Starting with

$$v_N(x,0)=P_N u(x,0)\,, \tag{1.5}$$

the classical Fourier method [3] lets $v_N(x,t)$ evolve at a later time according to the approximate model

$$\frac{\partial}{\partial t}v_N(x,t)+\frac{\partial}{\partial x}(\frac{1}{2}P_N v_N^2(x,t))=0\,. \tag{1.6}$$

Noting that P_N commutes with differentiation, we can rewrite (1.6) in the equivalent form

$$\frac{\partial}{\partial t}u_N(x,t) + \frac{\partial}{\partial x}(\frac{1}{2}u_N^2(x,t)) = (I - P_N)\frac{\partial}{\partial x}(\frac{1}{2}u_N^2(x,t)) \ . \tag{1.7}$$

Let us multiply (1.7) by $u_N(x,t)$: since $u_N(x,t)$ is orthogonal to the righthand-side of (1.7), we find after integration that

$$\frac{1}{2}\frac{d}{dt}\int_{-\pi}^{\pi} u_N^2(x,t)dx = -\int_{-\pi}^{\pi} u_N(x,t)\frac{\partial}{\partial x}(\frac{1}{2}u_N^2(x,t))dx = -\frac{u_N^3(x,t)}{3}\Big|_{x=-\pi}^{x=\pi} = 0 \ . \tag{1.8}$$

Thus $\int u_N^2(x,t)dx$ is conserved in time

$$\int_{-\pi}^{\pi} u_N^2(x,t)dx = \int_{-\pi}^{\pi} u_N^2(x,0)dx \le \int_{-\pi}^{\pi} u^2(x,0)dx \ , \tag{1.9}$$

and this yields the existence of a weak limit $\overline{u}(x,t) = \underset{N\to\infty}{w\lim} u_N(x,t)$. Does $\overline{u}(x,t)$ solves our problem? Unfortunately the answer is no. For otherwise, if $\overline{u}(x,t)$ is a weak solution of (1.1), then $P_N u_N^2(x,t)$ and hence $u_N^2(x,t)$ should tend weakly to $\overline{u}^2(x,t)$, and consequently, $\overline{u}(x,t)$ should be the *strong* limit of $u_N(x,t)$; but then (1.9) implies that $\int_{-\pi}^{\pi}\overline{u}^2(x,t)dx$ is also conserved in time, and by (1.2) this contradicts the appearance of physically relevant shock waves in our solution.

In practical applications, spectral methods are often augmented with smoothing procedures in order to give a helping hand toward their spectral convergence. Indeed, convergence for smoothed versions of spectral (and in particular pseudospectral) methods, was established in the linear case, e.g. [5], [7], [13]. However, arguments similar to the above show that with nonlinear problems, convergence of the Fourier method fails despite the additional smoothing of its solution. We leave the details for the appendix. Instead, we propose here a different way to enforce the convergence of the spectral-Fourier method without sacrifying spectral accuracy. This is accomplished by introducing, in Section 2, a new type of spectral vanishing viscosity. In Section 3 we prove the convergence of the proposed method using compensated compactness arguments, and in Section 4 it is shown that the limit solution respect the entropy condition (1.2). In Section 5 we extend our discussion to systems of conservation laws, and we show how the spectral vanishing viscosity can be used to enforce the correct entropy dissipation in such case. Finally, numerical

experiments with the proposed method of spectral regularisation are presented in Section 6.

2. The Spectral Vanishing Viscosity.

It is well known [6] that the unique entropy solution of (1.1), (1.2) is the one identified with the small viscosity limit of the regularized problem

$$\frac{\partial}{\partial t}u_\varepsilon(x,t) + \frac{\partial}{\partial x}(\frac{1}{2}u_\varepsilon^2(x,t)) = \varepsilon\frac{\partial}{\partial x}\left[Q\frac{\partial}{\partial x}u_\varepsilon(x,t)\right] , \qquad \varepsilon \downarrow 0 . \tag{2.1}$$

With the vanishing viscosity method [9], one replaces the exact derivatives in (2.1) by their discrete counterpart, the viscosity coefficient Q is chosen as (a nonlinear) positive grid dependent quantity, and the role of ε is played by some *fixed* power of the vanishing grid size, $\varepsilon \sim (\Delta x)^s$, in order to yield an s-order accurate approximation of (1.1). Yet in order to respect spectral accuracy, a more delicate viscous regularisation is required. To this end we consider viscosity coefficients of the form $Q = I - P_m$. The resulting viscosity terms are of spectrally small order of magnitude in the sense that for *any* $s > 0$ we have

$$\left\|\varepsilon\frac{\partial}{\partial x}\left[(I-P_m)\frac{\partial}{\partial x}u(x,t)\right]\right\|_{L^2(\Omega)} \le \varepsilon\|u\|_{H^{s+2}(\Omega)}\cdot m^{-s} ;$$

in fact these terms are exponentially small in the analytic case [10]. Together with this kind of spectral vanishing viscosity, the spectrally accurate Fourier approximation of (1.1) amounts to

$$\frac{\partial}{\partial t}u_N(x,t) + \frac{\partial}{\partial x}(\frac{1}{2}P_N u_N^2(x,t)) = \varepsilon\frac{\partial}{\partial x}\left[(I-P_m)\frac{\partial}{\partial x}u_N(x,t)\right] , \tag{2.2}$$

and we raise the question of its convergence as N tends to infinity. Here $\varepsilon \equiv \varepsilon(N) \downarrow 0$ and $m \equiv m(N) < N$ are free parameters which are yet to be determined, subject to the spectral accuracy restriction $m(N) \uparrow \infty$. In the next two sections we find such admissible parameters which provide a positive answer to the convergence question.

3. Spectral Convergence to a Weak Solution.

We consider the approximate Fourier method (2.2) which we rewrite as

$$\frac{\partial}{\partial t}u_N + \frac{\partial}{\partial x}(\frac{1}{2}u_N^2) = \frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2\right] + \varepsilon\frac{\partial}{\partial x}\left[(I-P_m)\frac{\partial}{\partial x}u_N\right] \equiv I + II\ . \tag{3.1}$$

In order to prove convergence of this method we need a couple of a' priori estimates on its solution. To this end, we multiply (3.1) by u_N,

$$\begin{aligned}\frac{\partial}{\partial t}(\frac{1}{2}u_N^2) + \frac{\partial}{\partial x}(\frac{1}{3}u_N^3) = \\ = u_N\frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2\right] + \varepsilon u_N\frac{\partial}{\partial x}\left[(I-P_m)\frac{\partial}{\partial x}u_N\right] \equiv III + IV\end{aligned} \tag{3.2}$$

and integrate over the 2π period: the integrals of the second and third terms vanish by periodicity and orthogonality, and we are left with

$$\frac{1}{2}\frac{d}{dt}\left\|u_N(\cdot,t)\right\|^2_{L^2(\Omega)} + \varepsilon\left\|(I-P_m)\frac{\partial}{\partial x}u_N(\cdot,t)\right\|^2_{L^2(\Omega)} = 0\ . \tag{3.3}$$

This gives us the a' priori bound on the amplitudes of the solution we had before in (1.9), and even a little more. More precisely, temporal integration of (3.3) yields

$$\frac{1}{2}\left\|u_N(\cdot,t)\right\|^2_{L^2(\Omega)} + \varepsilon\int_{\tau=0}^{t}\left\|(I-P_m)\frac{\partial}{\partial x}u_N(\cdot,\tau)\right\|^2_{L^2(\Omega)}d\tau = \frac{1}{2}\left\|u_N(\cdot,t=0)\right\|^2_{L^2(\Omega)}\ , \tag{3.4}$$

and hence for $u_N(x,t) \equiv \sum_{|k|\le N}\widehat{u}_k(t)e^{ikx}$ we have

$$\left\|u_N(\cdot,t)\right\|^2_{L^2(\Omega)} = \sum_{|k|\le N}|\widehat{u}_k(t)|^2 \le Const_1\ , \qquad Const_1 = \left\|u(\cdot,t=0)\right\|^2_{L^2(\Omega)}\ ; \tag{3.5}$$

equality (3.4) also gives us the second a' priori estimate

$$\varepsilon\left\|(I-P_m)\frac{\partial u_N}{\partial x}\right\|^2_{L^2_{loc}(\Omega,t)} \equiv \varepsilon\int_t\sum_{m<|k|\le N}k^2|\widehat{u}_k(t)|^2dt \le Const_2\ . \tag{3.6}$$

Equipped with these estimates we may turn now to the convergence proof of the Fourier method (2.2). We will establish spectral convergence for an admissible set of parameters $\varepsilon(N)\downarrow 0$, $m(N)\uparrow\infty$, using Tartar's div-curl lemma [14]. In order to apply the latter in our case, we have to verify that the four expressions appearing on the righthand-sides of (3.1) and (3.2) are 'nice' ones, namely, that these expressions are the sum of

terms, each of which lies either in the compact of $H^{-1}_{loc}(x,t)$, or - by Murat's lemma, in a bounded set of $L^1_{loc}(x,t)$. In the following lemmas we collect the necesary estimates in this direction.

We begin with the first term on the right of (3.1). Here, the following estimate whose proof is postponed to the end of this section, is essential.

Lemma 3.1. *There exists a constant $Const_{12}$ (depending on $Const_1$ and $Const_2$) such that for $m < \frac{1}{2}N$ we have*

$$\left\| \frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2\right] \right\|_{W^{-1}_{loc}(\pi,t)} \le \left\|(I-P_N)\frac{1}{2}u_N^2\right\|_{L^2_{loc}(\pi,t)} \le Const_{12}\cdot(\varepsilon N)^{-1/2}\,. \tag{3.7}$$

Next, we use the a' priori estimate (3.6) to conclude that, as ε tends to zero, the second term on the right of (3.1) belongs to the compact of $H^{-1}_{loc}(x,t)$.

Lemma 3.2. *The following estimate holds*

$$\begin{aligned} &\left\|\varepsilon\frac{\partial}{\partial x}\left[(I-P_m)\frac{\partial u_N}{\partial x}\right]\right\|_{W^{-1}_{loc}(\pi,t)} \le \\ &\le \varepsilon\left\|(I-P_m)\frac{\partial u_N}{\partial x}\right\|_{L^2_{loc}(\pi,t)} \le (\varepsilon\cdot Const_2)^{1/2}\to 0\,,\quad \varepsilon\downarrow 0\,. \end{aligned} \tag{3.8}$$

To treat the expressions on the right of (3.2), we first prepare

Lemma 3.3. *There exist constants $Const_3$ and $Const_{23}$ ($Const_{23}$ depending on $Const_2$ and $Const_3$) such that*

$$\left\|P_m\frac{\partial^s u_N}{\partial x^s}\right\|_{L^2_{loc}(\pi,t)} \le Const_3\cdot m^s \tag{3.9}$$

$$\left\|\frac{\partial u_N}{\partial x}\right\|_{L^2_{loc}(\pi,t)} \le Const_{23}\cdot(\varepsilon^{-1/2}+m) \tag{3.10}$$

Proof. The inequality (3.9) is immediate in view of $\left\|P_m\frac{\partial^s u_N}{\partial x^s}\right\|_{L^2_{loc}(\pi,t)} \le m^s\cdot\left\|u_N\right\|_{L^2_{loc}(\pi,t)}$ and the a' priori estimate (3.5). To prove (3.10) we invoke the identity

$$\left\|\frac{\partial u_N}{\partial x}\right\|^2_{L^2_{loc}(\pi,t)} = \left\|(I-P_m)\frac{\partial u_N}{\partial x}\right\|^2_{L^2_{loc}(\pi,t)} + \left\|P_m\frac{\partial u_N}{\partial x}\right\|^2_{L^2_{loc}(\pi,t)}\,,$$

and use the a' priori estimate (3.6) to upper bound the first norm, and (3.9) for the second one.

In the next two lemmas we turn to deal with the righthand-side of (3.2). For its first member - which we express as

$$
\begin{aligned}
III &\equiv u_N \frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2\right] = \\
&= \frac{\partial}{\partial x}\left[u_N(I-P_N)\frac{1}{2}u_N^2\right] - \frac{\partial u_N}{\partial x}(I-P_N)\frac{1}{2}u_N^2 \equiv III_1 + III_2
\end{aligned}
\tag{3.11}
$$

we have

Lemma 3.4. *There exists a constant $Const_{123}$ (depending on $Const_{12}$ and $Const_{23}$) such that the following estimates hold.*

$$
\|III_1 \equiv \frac{\partial}{\partial x}\left[u_N(I-P_N)\frac{1}{2}u_N^2\right]\|_{W^{-1}_{l^\infty}(x,t)} \le Const_{123}\cdot\|u_N\|_{L^\infty(x,t)}\cdot(\varepsilon N)^{-1/2}\ , \tag{3.12}
$$

$$
\|III_2 \equiv -\frac{\partial u_N}{\partial x}(I-P_N)\frac{1}{2}u_N^2\|_{L^1_{l^\infty}(x,t)} \le Const_{123}\cdot(\varepsilon^{-1/2}+m)\cdot(\varepsilon N)^{-1/2}\ . \tag{3.13}
$$

Proof. The first estimate, (3.12), follows from Lemma 3.1 with $Const_{123} = Const_{12}$

$$
\begin{aligned}
\|III_1\|_{W^{-1}_{l^\infty}(x,t)} &\le \|u_N(I-P_N)\frac{1}{2}u_N^2\|_{L^2_{l^\infty}(x,t)} \le \\
&\le \|u_N\|_{L^\infty(x,t)}\cdot\|(I-P_N)\frac{1}{2}u_N^2\|_{L^2_{l^\infty}(x,t)} \le \\
&\le Const_{12}\cdot\|u_N\|_{L^\infty(x,t)}(\varepsilon N)^{-1/2}\ ;
\end{aligned}
$$

together with Lemma 3.3 we also have

$$
\|III_2\|_{L^1_{l^\infty}(x,t)} \le \|\frac{\partial u_N}{\partial x}\|_{L^2_{l^\infty}(x,t)}\cdot\|(I-P_N)\frac{1}{2}u_N^2\|_{L^2_{l^\infty}(x,t)}\ ,
$$

and the second estimate, (3.13), follows with $C_{123} = Const_{12}\cdot Const_{23}$.

Finally, the second member on the right of (3.2),

$$
\begin{aligned}
IV &\equiv \varepsilon u_N(I-P_m)\frac{\partial^2 u_N}{\partial x^2} = \\
&= \varepsilon\frac{\partial}{\partial x}\left[u_N(I-P_m)\frac{\partial u_N}{\partial x}\right] - \varepsilon\frac{\partial u_N}{\partial x}(I-P_m)\frac{\partial u_N}{\partial x} \equiv IV_1 + IV_2\ ,
\end{aligned}
\tag{3.14}
$$

is estimated as follows.

Lemma 3.5. *There exists a constant $Const_{32}$ (depending on $Const_2$ and $Const_3$) such that the following estimates hold.*

$$\|IV_1 \equiv \varepsilon \frac{\partial}{\partial x}\left[u_N(I-P_m)\frac{\partial u_N}{\partial x}\right]\|_{W^{-1}_{loc}(\Omega,t)} \leq Const_{32}\cdot\|u_N\|_{L^\infty(\Omega,t)}\cdot\varepsilon^{1/2} \tag{3.15}$$

$$\|IV_2 \equiv -\varepsilon\frac{\partial u_N}{\partial x}(I-P_m)\frac{\partial u_N}{\partial x}\|_{L^1_{loc}(\Omega,t)} \leq Const_{32}(1+\varepsilon^{1/2}m)\ . \tag{3.16}$$

Proof. The first inequality, (3.16), follows from the a' priori estimate (3.6),

$$\|IV_1\|_{W^{-1}_{loc}(\Omega,t)} \leq \varepsilon\|u_N\|_{L^\infty(\Omega,t)}\cdot\|(I-P_m)\frac{\partial u_N}{\partial x}\|_{L^2_{loc}(\Omega,t)} \leq \|u_N\|_{L^\infty(\Omega,t)}\cdot(Const_2\cdot\varepsilon)^{1/2}\ .$$

To prove the second inequality we upper bound

$$\|IV_2\|_{L^1_{loc}(\Omega,t)} \leq \varepsilon\|(I-P_m)\frac{\partial u_N}{\partial x}\|^2_{L^2_{loc}(\Omega,t)} + \sqrt{\varepsilon}\|P_m\frac{\partial u_N}{\partial x}\|_{L^2_{loc}(\Omega,t)}\cdot\sqrt{\varepsilon}\|(I-P_m)\frac{\partial u_N}{\partial x}\|_{L^2_{loc}(\Omega,t)},$$

and use the a' priori estimates (3.6) and (3.9) to obtain (3.16) with $Const_{32} = Const_2\cdot(Const_3+1)$.

We are now ready to find the admissible parameters which meet the assumptions of the div-curl lemma. By (3.16), the term IV_2 is bounded in $L^1_{loc}(x,t)$ if $\varepsilon^{1/2}m \leq Const.$; choosing $\varepsilon \sim \frac{Const.}{m^2}$, then by (3.13) the term III_2 is bounded in $L^1_{loc}(x,t)$ provided $m^2N^{-1/2} \leq Const$; choosing $m \leq \min\,(Const.N^{1/4},\ \frac{1}{2}N)$ we conclude by Lemma 3.1 and Lemma 3.2, that the terms I and II belong to the compact of $H^{-1}_{loc}(x,t)$. Moreover, if u_N is uniformly bounded,

$$\|u_N\|_{L^\infty(\Omega,t)} \leq Const.\ , \tag{3.17}$$

then by Lemmas 3.4 and 3.5 we have with this choice of parameters that the terms III_1 and IV_1 are also in the compact of $H^{-1}_{loc}(x,t)$. This completes our study of the expressions on the right of (3.1), (3.2) and the div-curl lemma applies in our case. We summarize by stating

Theorem 3.6. *Consider the spectral approximation (2.2) with parameters (ε, m) which satisfy*

$$\varepsilon(N) \sim Const.N^{-2\beta}\ , \quad m = m(N) \sim Const.N^{\beta}\ , \qquad 0 < \beta \leq \frac{1}{4}\ . \tag{3.18}$$

Assume that its solution, $u_N(x,t)$, remains uniformly bounded, (3.17). Then $u_N(x,t)$ converges boundedly a.e. to a weak solution of the conservation law (1.1)

Proof. Let $\overline{u^{(j)}}(x,t)$ denote the weak limit of $u_N^j(x,t)$

$$\underset{N\to\infty}{w\lim}\ u_N^j(x,t) = \overline{u^{(j)}}(x,t)\ .$$

Applying the div-curl lemma to (3.1), (3.2) we have the relation

$$\overline{u^{(4)}} = 4\overline{u^{(1)}}\cdot\overline{u^{(3)}} - 3(\overline{u^{(2)}})^2\ , \tag{3.19}$$

which implies strong convergence. To establish this implication, we follow the argument of Tartar [15] (see also [14]), who suggests to consider the weak limit of $(u_N - \overline{u^{(1)}})^4$,

$$\underset{N\to\infty}{w\lim}\ (u_N - \overline{u^{(1)}})^4 = \overline{u^{(4)}} - 4\overline{u^{(3)}}\ \overline{u^{(1)}} + 6\overline{u^{(2)}}(\overline{u^{(4)}})^2 - 4(\overline{u^{(1)}})^3 + (\overline{u^{(1)}})^4\ ;$$

using (3.19) and rearranging, we find

$$\underset{N\to\infty}{w\lim}\ (u_N - \overline{u^{(1)}})^4 = -3\left[\overline{u^{(2)}} - (\overline{u^{(1)}})^2\right]^2 \leq 0\ ,$$

and hence $\overline{u^{(2)}} = (\overline{u^{(1)}})^2$. Consequently, $u_N(x,t)$ converges strongly to $\overline{u}(x,t) \equiv \overline{u^{(1)}}(x,t)$ in $L^p_{loc}(x,t)$, and by (3.1), $\overline{u}(x,t)$ is a weak solution of (1.1).

We do not claim that our parametrization (3.18) is optimal. In particular, the restrictive choice of $m(N)$ could be improved as indicated by the numerical tests described in Section 6. On the other hand, we note that the ε parametrization (3.18) such that $\varepsilon m^2 \sim Const.$ yields, in view of Lemma (3.10),

$$\sqrt{\varepsilon}\left\|\frac{\partial u_N}{\partial x}\right\|_{L^2_{loc}(x,t)} \leq Const.\ , \tag{3.20a}$$

which is in complete agreement with the behavior of the viscous regularization model (2.1), where

$$\sqrt{\varepsilon}\left\|\frac{\partial u_\varepsilon}{\partial x}\right\|_{L^2_{loc}(x,t)} \leq Const. \tag{3.20b}$$

Our choice of parameters in (3.18) depends heavily on the essential estimate (3.7), and we conclude this section with its proof.

Proof. (of Lemma 3.1.) We should upper bound the norm of

$$\left\|(I-P_N)\frac{1}{2}u_N^2\right\|^2_{L^2_{loc}(x,t)} = \tag{3.21}$$

$$= \frac{1}{2}\int_t \sum_{N<p\le 2N} \Big|\sum_{k=p-N}^{N} \widehat{u}_k(t)\overline{\widehat{u}}_{p-k}(t)\Big|^2 + \frac{1}{2}\int_t \sum_{-2N<p\le -N} \Big|\sum_{k=-N}^{N+p} \widehat{u}_k(t)\overline{\widehat{u}}_{p-k}(t)\Big|^2 .$$

The first intergral on the right does not exceed

$$\frac{1}{2}\int_t \sum_{N<p\le 2N}\left[\left(\sum_{k=p-N}^{p/2}+\sum_{k=p/2}^{N}\right)|\widehat{u}_k(t)|\cdot|\widehat{u}_{p-k}(t)|\right]^2 =$$

$$= 2\int_t \sum_{N<p\le 2N}\left[\sum_{k=p/2}^{N}|\widehat{u}_k(t)|\cdot|\widehat{u}_{p-k}(t)|\right]^2 ,$$

and using Cauchy-Schwartz inequality this is less than

$$2\int_t \sum_{N<p\le 2N}\left(\sum_{k\ge p/2}^{N} k^2|\widehat{u}_k(t)|^2\cdot\sum_{k\ge p/2}^{N}\frac{1}{k^2}|\widehat{u}_{p-k}(t)|^2\right) .$$

According to our assumption $m<\frac{1}{2}N$. Hence for $p>N$ we have $p/2>m$, and by the a' priori estimates (3.5), (3.6), the last expression is bounded from above by

$$2\cdot \underset{t}{Max}\left(\sum_{N<p\le 2N}\frac{4}{p^2}\cdot\sum_k|\widehat{u}_{p-k}(t)|^2\right)\cdot\int_t\sum_{k>m}^{N}k^2|\widehat{u}_k(t)|^2$$

$$\le 2\cdot\sum_{N<p\le 2N}\frac{4}{p^2}\cdot \underset{t}{Max}\left\|u_N(\cdot,t)\right\|^2_{L^2(x)}\cdot\left\|(I-P_m)\frac{\partial u_N}{\partial x}\right\|^2_{L^2_{loc}(x,t)} \le \frac{8}{N}\cdot Const_1\cdot\frac{Const_2}{\epsilon}.$$

The second int gral on the right of (3.21) can be treated similarily and Lemma 3.1 follows with $Const_{12}=4\cdot(Const_1\cdot Const_2)^{1/2}$.

4. Spectral Convergence to the Entropy Solution.

In the last sections we have seen that the spectral approximation, $u_N(x,t)$, has a strong limit, $L^2_{loc}- \lim_{N\to\infty} u_N(x,t) = \overline{u}(x,t)$, which is a weak solution of (1.1). In this section we show tha. .his limit is in fact the unique entropy solution of (1.1) satisfying the entropy inequality (1.2).

Theorem 4.1. *Consider the spectral approximation (2.2) with parameters (ε, m) which satisfy*

$$\varepsilon = \varepsilon(N) \sim Const.N^{-2\beta}, \quad m = m(N) \sim Const.N^{\beta}, \qquad 0 < \beta < 1/4 \,. \tag{4.1}$$

Assume that its solution, $u_N(x,t)$ remains uniformly bounded, (3.17). Then $u_N(x,t)$ converges boundedly a.e. to the unique entropy solution of the conservation law (1.1).

Proof. Consider the righthand-side of (3.2) which consists of the sum of two terms, III + IV. We will show that this sum tends weakly to a negative measure and hence convergence to the entropy inequality (1.2) follows.

As in (3.11) we write $III = III_1 + III_2$, where by Lemma 3.4

$$\left\|III_1\right\|_{M^{-1}_{loc}(\pi,\tau)} \leq Const.\left\|u_N\right\|_{L^\infty(\pi,\tau)} \cdot N^{\beta-1/2} \to 0 \,, \tag{4.2}$$

and in view of our slightly strengthened parametrization (5.1) (compared with (3.18)),

$$\left\|III_2\right\|_{L^1_{loc}(\pi,\tau)} \leq Const.N^{2\beta-1/2} \to 0 \,, \qquad 0 < \beta < 1/4 \,. \tag{4.3}$$

Consequently, the first term on the right of (3.2), III, tends weakly to zero, and we turn to deal with the second one which is given in (3.14) as $IV = IV_1 + IV_2$. By Lemma 3.5 we have

$$\left\|IV_1\right\|_{M^{-1}_{loc}(\pi,\tau)} \leq Const.\left\|u_N\right\|_{L^\infty(\pi,\tau)} \cdot N^{-\beta} \to 0 \,, \tag{4.4}$$

and hence the term IV_1 also tends weakly to zero. Finally we are left with the term $IV_2 = -\varepsilon\frac{\partial u_N}{\partial x}(I - P_m)\frac{\partial u_N}{\partial x}$, which we write as

$$IV_2 = -\varepsilon(\frac{\partial u_N}{\partial x})^2 + \varepsilon\frac{\partial}{\partial x}[u_N P_m \frac{\partial u_N}{\partial x}] - \varepsilon u_N P_m \frac{\partial^2 u_N}{\partial x^2} \equiv IV_{21} + IV_{22} + IV_{23} \,. \tag{4.5}$$

it follows from (3.20a), that $-\varepsilon(\frac{\partial u_N}{\partial x})^2$ tends weakly to a negative measure,

$$\underset{N\to\infty}{w\lim}\ [IV_{21} \equiv (\frac{\partial u_N}{\partial x})^2] \le 0\,. \tag{4.6}$$

Also, the pessimistic bound

$$\|-\varepsilon\frac{\partial}{\partial x}[u_N P_m \frac{\partial u_N}{\partial x}]\|_{H^{-1}_{loc}(\pi,t)} \le \varepsilon\cdot\|u_N\|_{L^\infty(\pi,t)}\cdot\|P_m\frac{\partial u_N}{\partial x}\|_{L^2_{loc}(\pi,t)}\,,$$

yields by Lemma 3.3,

$$\|IV_{22} \equiv \varepsilon\frac{\partial}{\partial x}[u_N P_m \frac{\partial u_N}{\partial x}]\|_{H^{-1}_{loc}(\pi,t)} \le Const.\|u_N\|_{L^\infty(\pi,t)}\cdot N^{-\beta}\to 0\,, \tag{4.7}$$

and hence weak convergence to zero. We conclude with

$$IV_{23} \equiv -\varepsilon u_N P_m\frac{\partial^2 u_N}{\partial x^2} = -\varepsilon\overline{u}P_m\frac{\partial^2 u_N}{\partial x^2} - \varepsilon(u_N-\overline{u})P_m\frac{\partial^2 u_N}{\partial x^2} \equiv IV_{231}+IV_{232}\,. \tag{4.8}$$

Here we have

$$IV_{231} \equiv -\varepsilon\overline{u}P_m\frac{\partial^2 u_N}{\partial x^2}\to 0 \qquad \text{(weakly)}, \tag{4.9}$$

and by Lemma 3.3.

$$\begin{aligned}\|IV_{232} \equiv -\varepsilon(u_N-\overline{u})P_m\frac{\partial^2 u_N}{\partial x^2}\|_{L^1_{loc}(\pi,t)} \le\\ \le \varepsilon\cdot\|u_N-\overline{u}\|_{L^2_{loc}(\pi,t)}\cdot\|P_m\frac{\partial^2 u_N}{\partial x^2}\|_{L^2_{loc}(\pi,t)} \le\\ \le Const.N^{-2\beta}\cdot N^{2\beta}\cdot\|u_N-\overline{u}\|_{L^2_{loc}(\pi,t)}\to 0\,.\end{aligned} \tag{4.10}$$

In summary, we have by (4.2)-(4.10) that the righthand-side tends in the sense of distributions to a negative measure, while the (weak) limit on the left gives us

$$\frac{\partial}{\partial t}(\frac{1}{2}\overline{u}^2)+\frac{\partial}{\partial x}(\frac{1}{3}\overline{u}^3)\le 0\,,$$

as asserted.

5. Systems of Conservation Laws.

It is instructive to compare between the spectral methods before and after spectral vanishing viscosity was added. Before viscosity was added we had, in (1.6), a coupled system of O.D.E's in the Fourier space, which amounts to

$$\frac{d}{dt}\hat{u}_k(t) + \frac{1}{2}ik \sum_{p+q=k} \hat{u}_p(t)\hat{u}_q(t) = 0 \qquad |k| \le N \ . \tag{5.1}$$

In this case the total quadratic entropy was conserved, (1.9), which is responsible to the divergence of the method. After viscosity was added in (2.2), the resulting system in the Fourier space reads

$$\frac{d}{dt}\hat{u}_k(t) + \frac{1}{2}ik \sum_{p+q=k} \hat{u}_p(t)\hat{u}_q(t) = 0 \qquad |k| \le m \ . \tag{5.2a}$$

$$\frac{d}{dt}\hat{u}_k(t) + \frac{1}{2}ik \sum_{p+q=k} \hat{u}_p(t)\hat{u}_q(t) = -\varepsilon k^2 \hat{u}_k(t) \qquad |k| > m \ . \tag{5.2b}$$

An increasing portion of the spectrum is treated here as in the diverging case (5.1). Yet, the added viscosity for the high Fourier modes in (5.2b) is responsible for the correct rate of entropy dissipation (3.6) which in turn implies convergence in the scalar case. In this section we show how to enforce similar entropy dissipation by spectral vanishing viscosity in systems of conservation laws. To this end we proceed as follows.

Consider the conservative system

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\left[f(u(x,t))\right] = 0 \ , \tag{5.3}$$

which is assummed to be equipped with an entropy function $U(u)$, i.e., an convex function whose Hessian U_{uu} symmetrizes the Jacobian matrix f_u, e.g. [2], [11]. Using the entropy variables

$$v = v(u) = \frac{\partial U}{\partial u}(u) \ , \tag{5.4a}$$

the conservative system (5.3) takes the equivalent symmetric form, consult [4], [8], [12]

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}[g(v(x,t))] = 0 \ , \qquad g(v) \equiv f(u(v)) \ . \tag{5.4b}$$

The Fourier approximation of (5.3) will be based on this formulation: together with additional vanishing spectral viscosity we arrive at

$$\frac{\partial}{\partial t}u_N(x,t) + \frac{\partial}{\partial x}[P_N g(v_N(x,t))] = \varepsilon \frac{\partial}{\partial x}\left[(I - P_m)\frac{\partial}{\partial x}v_N(x,t)\right] \tag{5.5a}$$

where $v_N(x,t)$ is the projected vector of entropy variables

$$v_N(x,t) = P_N[v(u_N(x,t))] \tag{5.5b}$$

Multiply (5.5a) by $v_N(x,t)$ and integrate over the 2π-period: taking into account orthogonality we have

$$\frac{d}{dt}\int_{-\pi}^{\pi} U(u_N(x,t))dx + \int_{-\pi}^{\pi} v_N(x,t)\frac{\partial}{\partial x}[g(v_N(x,t)]dx$$
$$= \varepsilon \int_{-\pi}^{\pi} v_N(x,t)\frac{\partial}{\partial x}[(I - P_m)\frac{\partial}{\partial x}v_N(x,t)]dx \,.$$

The second integrand on the left is a perfect derivative of the associated entropy flux and hence its integral vanishes. Integration by parts on the right yields

$$\frac{d}{dt}\int_{-\pi}^{\pi} U(u_N(x,t))dx + \varepsilon\|(I - P_m)\frac{\partial}{\partial x}v_N(\cdot,t)\|^2_{L^2(x)} = 0 \,, \tag{5.6}$$

which shows that entropy dissipates at the correct rate. In particular, arguing along the lines of Lemma 3.3 and using the strict convexity of $U(u)$, we conclude that

$$\sqrt{\varepsilon}\|\frac{\partial u_N}{\partial x}\|_{L^2_{loc}(x,t)} \leq Const. \,, \qquad \varepsilon m^2 \sim Const. \tag{5.7}$$

This is analogous to the behavior of the viscous regularization for (5.3) - compare (3.20) in the scalar case.

6. Numerical Experiments.

The Fourier method with spectral vanishing viscosity was applied to the periodic Burgers' equation (1.1) with $u(x,t=0) = \sin x$ as initial data. The resulting O.D.E. system for the Fourier coefficients, see (5.2)

$$\frac{d}{dt}\hat{u}_k(t) + \frac{1}{2}ik \sum_{p+q=k} \hat{u}_p(t)\hat{u}_q(t) = -\varepsilon k^2 \hat{Q}(k)\hat{u}_k(t) \ , \quad |k| \le N \ , \tag{6.1a}$$

$$\hat{Q}(k) = \begin{cases} 0 & |k| \le m \ , \\ 1 & |k| > m \ , \end{cases} \tag{6.1b}$$

was integrated up to time $t = 1.5$, using the fourth-order Runge-Kutta method.

The number of significant modes was chosen as a fraction of the total number, $m = \theta N$. The numerical experiments indicate, as expected, that the quality of the results is more sensitive to the dependence of ε on m; further investigation is necessary in order to exhaust this point. In the following examples we have, $\varepsilon m \sim 0.25$, based on considerations of minimising the total-variation of the numerical solution. With this choice of parameters, Figure 6.1 shows that the numerical solution converges strongly (but not uniformly) to the entropy solution of (1.1). This is in sharp constrast to the oscillatory behavior of the viscosity-free Fourier method in Figure 6.2, where $\hat{Q}(k) \equiv 0$*. Other parametarisations of ε, quoted in Figures 6.3 and 6.4, demonstrate the sensitivity of the computed solution mentioned earlier.

To improve the quality of these results, the proposed method (6.1a) was implemented with a spectral vanishing viscosity $\hat{Q}(k)$ which is *smoothly* varying between zero and one, say, for $\frac{m}{2} \le |k| < m$. Figures 6.5 and 6.6 show that this kind of viscosity prevents the propagation of the Gibbs phenomenan into the whole computational domain that was noticed earlier. This is analogous to the spectral recovery in shadowed regions between propagating *linear* discontinuties described in [7].

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* In fact, a slight amount of dissipation was introduced in this case due to the time integrator.

The Fourier method with spectral vanishing viscosity ... and without spectral viscosity.

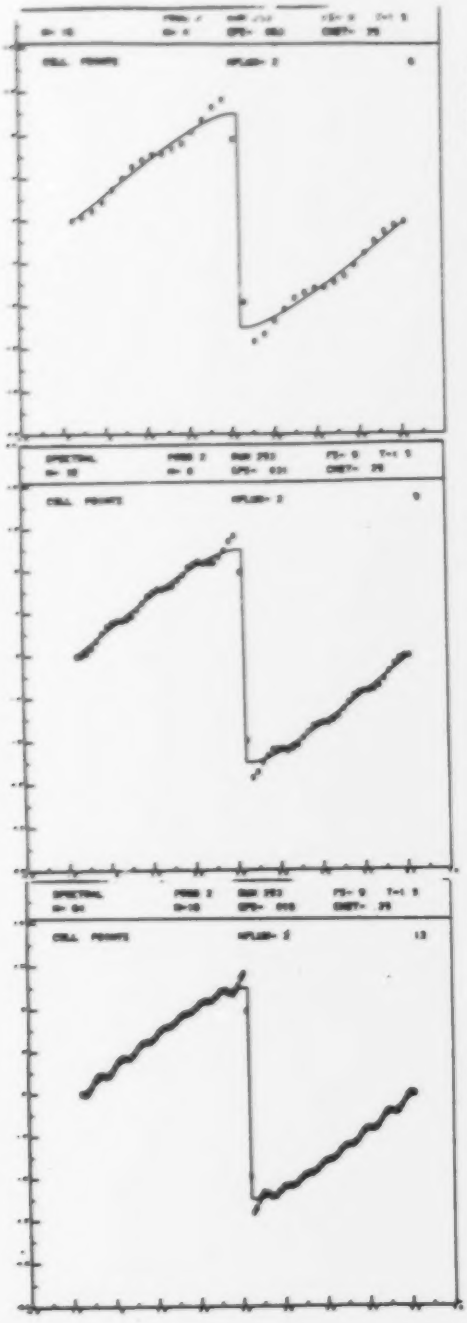


Figure 6.1.

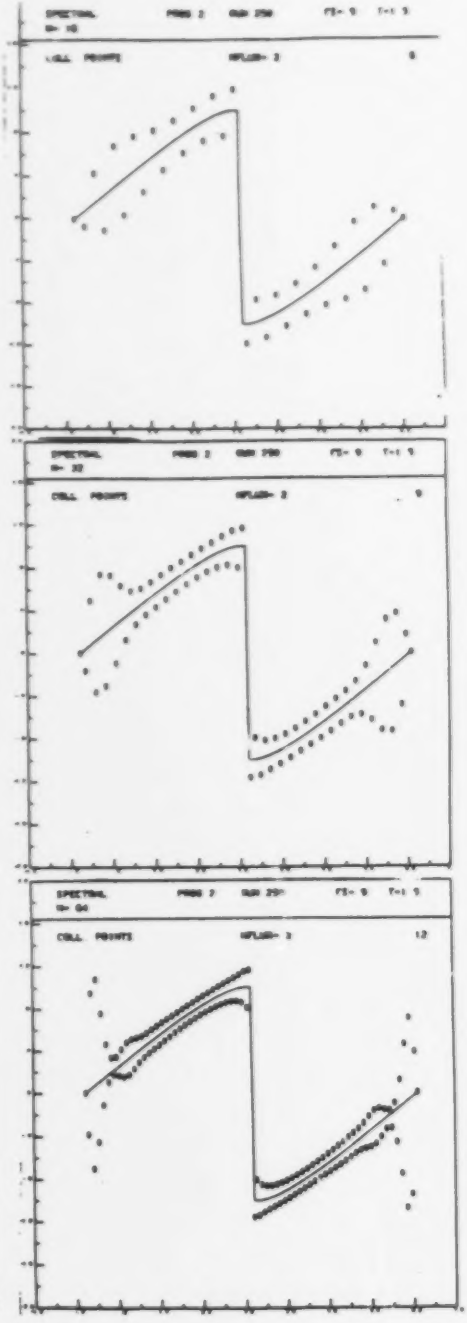
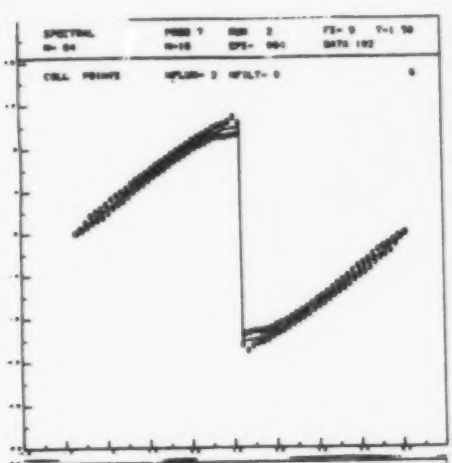
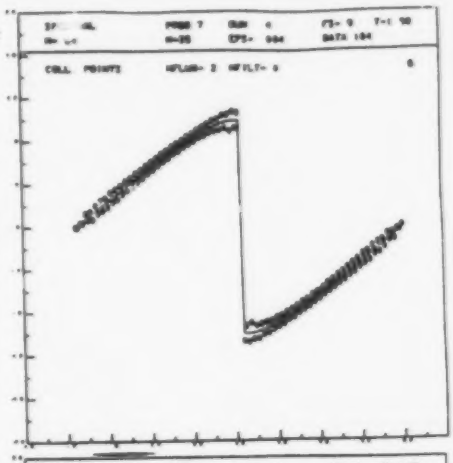
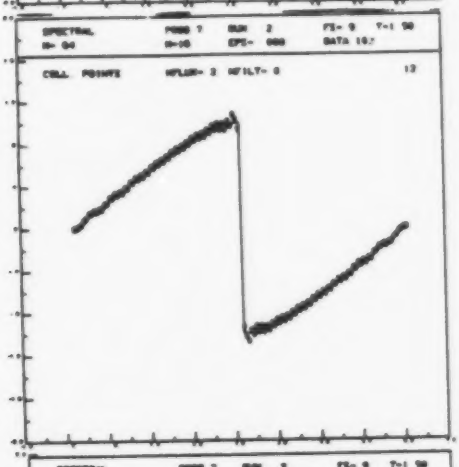


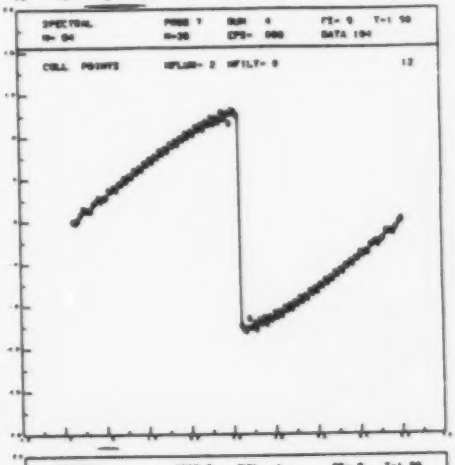
Figure 6.2.

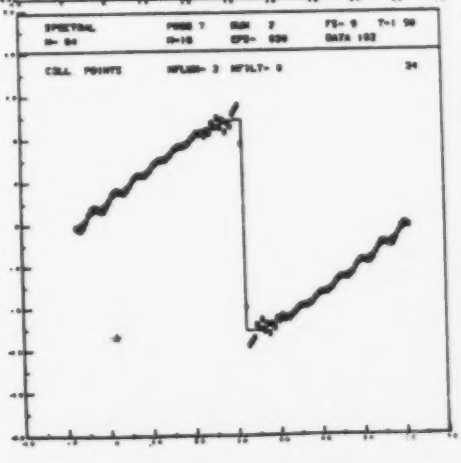
The Fourier method with various parameterisations of spectral vanishing viscosity.











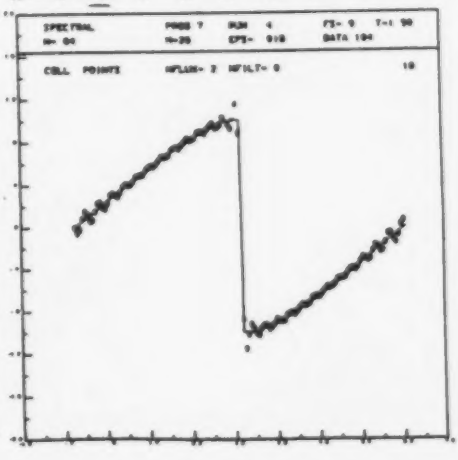


Figure 6.3.

Figure 6.4.

The Fourier method with smooth spectral vanishing viscosity.

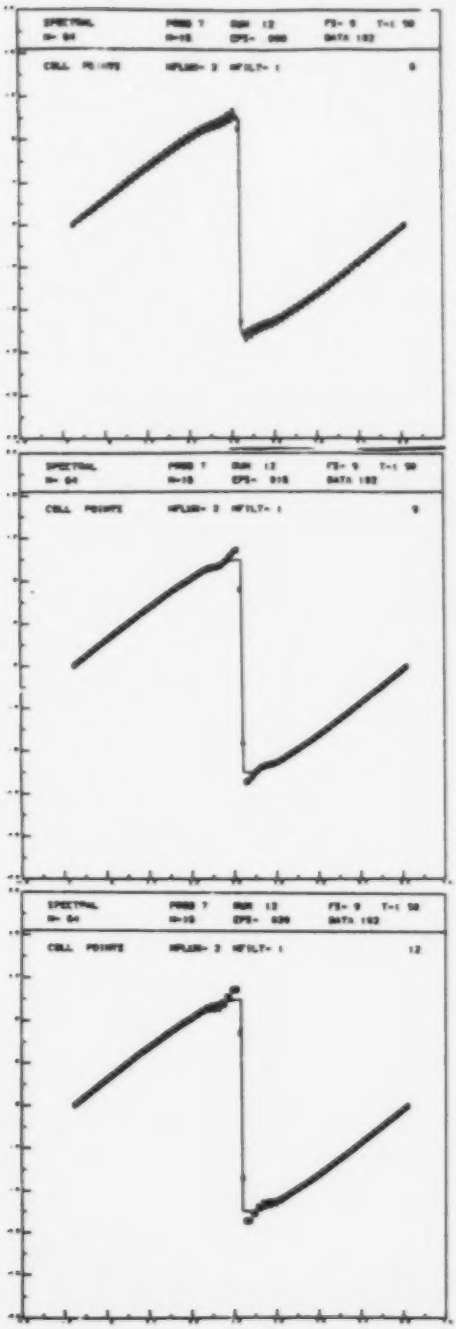


Figure 6.5.

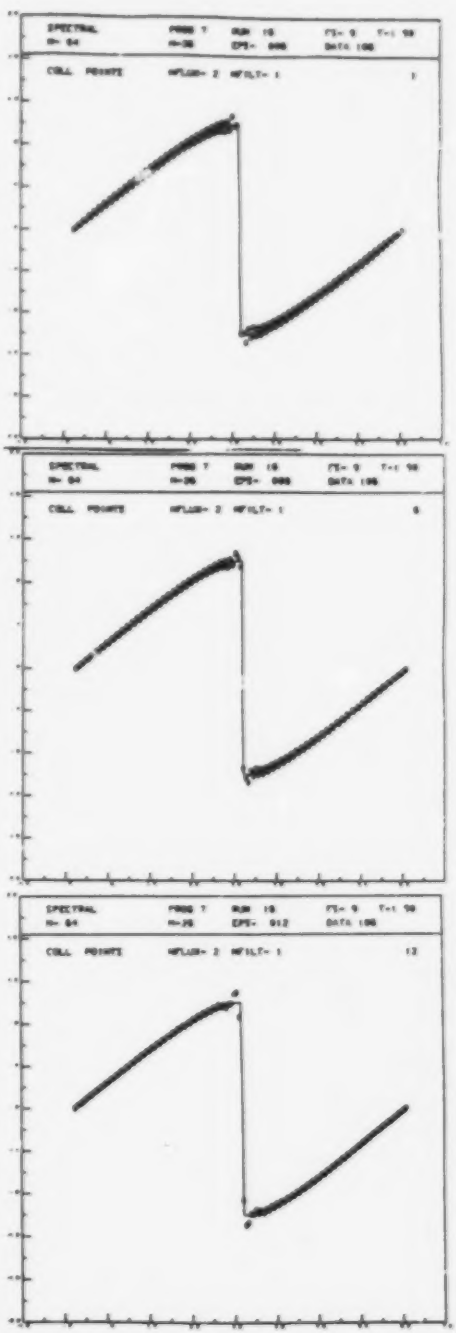


Figure 6.6.

APPENDIX

A. Failure of Convergence with Post-Processing.

In practical applications the classical Fourier method is often coupled with certain smoothing procedures whose purpose is to gain spectral convergence that otherwise might be lost. In a typical case, the solution is post-processed via a convolution with a smoothing kernel $Q_N \equiv Q_N(x)$

$$Q_N * u_N(x,t) = \sum_{k=-N}^{N} \widehat{q}_k \widehat{u}_k(t) e^{ikx} , \qquad 0 \le \widehat{q}_k \le \widehat{q}_0 = 1 , \qquad \widehat{q}_k \equiv \widehat{q}_{k,N} .$$

In order to maintain spectral accuracy, the convolution with such smoothing kernel should be highly accurate with that of Dirac's δ-distribution. We shall make a minimal assumption in this direction, requiring that for all square-integrable functions $\varphi(x)$ we have

$$\text{(a.2)} \qquad \|(\delta - Q_N) * \varphi(x)\|_{L^\infty(x)} \le \sum_{k=-N}^{N} |1 - \widehat{q}_{k,N}| \cdot |\widehat{\varphi}(k)| \underset{N\to\infty}{\to} 0 .$$

Such smoothing procedure enables, for example, spectral recovery of solutions to *linear* hyperbolic problems in the presence of propagating initial discontinuities, e.g. [1], [7].

We will show that the smoothed version of the Fourier approximation to the *nonlinear* inviscid Burgers' equation (1.1),

$$\text{(a.3)} \qquad \frac{\partial}{\partial t} u_N(x,t) + \frac{\partial}{\partial x}[\frac{1}{2} P_N (Q_N * u_N(x,t))^2] = 0 ,$$

where the *solution* (rather than the flux) is convolved with kernels satisfying (a.2), does not converge to the entropy solution of (1.1).

To this end we convolve (a.3) with Q_N to find that $w_N(x,t) = Q_N * u_N(x,t)$ satisfies

$$\text{(a.4)} \qquad \frac{\partial}{\partial t} w_N(x,t) + Q_N * \frac{\partial}{\partial x}[\frac{1}{2} P_N w_N^2(x,t)] = 0 .$$

Multiplying (a.4) by $u_N(x,t)$ and integrating over the 2π-period we obtain

$$\int_x u_N \frac{\partial w_N}{\partial t} dx = - \int_x Q_N * u_N \frac{\partial}{\partial x}[\frac{1}{2} P_N w_N^2] dx ,$$

or, in view of orthogonality

$$\frac{1}{2}\frac{d}{dt}\int_x u_N Q_N * u_N dx = -\int_x w_N \frac{\partial}{\partial x}[\frac{1}{2}w_N^2]dx = 0\ .$$

Thus with $Q_N^{1/2} * \varphi \equiv \sum_{k=-N}^{N} \widehat{q}_k^{1/2} \cdot \hat{\varphi}(k)$ we have

$$\|Q_N^{1/2} * u_N(\cdot,t)\|^2_{L^2(x)} = \sum_{k=-N}^{N} \widehat{q}_k |\widehat{u}_k(t)|^2 \equiv Const. \tag{a.5}$$

This implies that our smoothed approximation $w_N(x,t) = Q_N * u_N(x,t)$ converges to a weak limit $\underset{N\to\infty}{w\lim} w_N(x,t) = \overline{w}(x,t)$. Now, suppose that $\overline{w}(x,t)$ is a weak solution of (1.1); then this will lead us to the conservation of $\int_x \overline{w}^2(x,t)dx$ which shows that $\overline{w}(x,t)$ is *not* the entropy solution (1.1). Indeed, if $\overline{w}(x,t)$ satisfies (1.1) then by (a.4), $Q_N * P_N w_N^2(x,t)$ tends weakly to $\overline{w}^2(x,t)$

$$\underset{N\to\infty}{w\lim} Q_N * P_N w_N^2(x,t) = \overline{w}^2(x,t)\ . \tag{a.6}$$

It follows from (a.5) that $\|P_N w_N^2(\cdot,t)\|_{L^1(x)}$ is bounded, and together with (a.2) this implies that for all C_0^∞-test functions $\varphi(x)$ we have

$$\int_x \varphi(x)(\delta - Q_N) * P_N w_N^2(x,t)dx \le Const.\|(\delta - Q_N) * \varphi(x)\|_{L^\infty(x)} \underset{N\to\infty}{\to} 0\ . \tag{a.7}$$

Adding this to (a.6), we conclude that $P_N w_N^2(x,t)$ and hence $w_N^2(x,t)$ tend weakly to $\overline{w}^2(x,t)$. Consequently $\overline{w}(x,t)$ is the *strong* limit of our smoothed approximation

$$s\lim_{N\to\infty} Q_N * u_N(x,t) = \overline{w}(x,t)\ . \tag{a.8}$$

Finally, in view of (a.5) we can apply (a.2) to find that

$$\|Q_N * u_N(x,t) - Q_N^{1/2} * u_N(x,t)\|_{L^\infty(x)} \le 2\sum_{q=N}^{N} |1 - \widehat{q}_{k,N}| \cdot \widehat{q}_{k,N} \cdot |\widehat{u}_k(t)| \underset{N\to\infty}{\to} 0\ . \tag{a.9}$$

From (a.8) and (a.9) it follows that $\overline{w}(x,t)$ is the *strong* limit of $Q_N^{1/2} * u_N(x,t)$, and the strong limit of (a.5) tells us that $\int_x \overline{w}^2(x,t)dx$ is conserved in time which completes our asserted contradiction. We summarize by stating

Theorem A.1. ***The Fourier method (a.3) which employs any smoothing kernel satisfying (a.2), does not converge to the entropy solution of (1.1), (1.2).***

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Final Report

16. Abstract

We discuss the convergence of Fourier method for scalar nonlinear conservation laws which exhibit spontaneous shock discontinuities. Numerical tests indicate that the convergence may (and in fact in some cases must) fail, with or without post-processing of the numerical solution. Instead, we introduce here a new kind of spectrally accurate vanishing viscosity to augment the Fourier approximation of such nonlinear conservation laws. Using compensated compactness arguments, we show that this spectral viscosity prevents oscillations and convergence to the unique entropy solution follows.

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